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CUTSET CONDITION FOR BOOLEAN LATTICES

JUN WU

Department of Applied Mathematics
Dalian University of Technology, Dalian 116024, P. R. China
e-mail: junwue@gmail.com

Abstract. This paper proves a geometric lattice of finite length is a Boolean lattice if each of its maximal chains contains a neutral element different from its maximal element and minimal element. This result is generalized to atomistic lattices of finite length.

1. Introduction

Let L be a lattice with *meet* $\wedge$, *join* $\vee$, minimal element $\hat{0}$ and maximal element $\hat{1}$. Define the *length* of L, denote by $\ell(L)$, to be the least upper bound of the lengths of chains in L. Throughout this paper L always is a lattice of finite length, that is, $\ell(L)$ is finite. For $a, b \in L$, we say a covers b or b is covered by a, written as $a \prec b$, if there is no element $c \in L$ such that $b < c < a$. An element $p \in L$ is an *atom* of L if p covers $\hat{0}$. A lattice L is called an atomistic lattice if every element ($\neq \hat{0}$) is a join of atoms in L. We say L is an (*upper*) *semimodular* lattice if for any two distinct elements $a, b \in L$, both a and b covers $a \wedge b$ implies $a \vee b$ covers both a and b. An atomistic lattice L is a *geometric* lattice if L is semimodular.

A pair of elements (a, b) of L is called a *modular pair*, written as aMb, if for every $c \leq b$, $c \vee (a \wedge b) = (c \vee a) \wedge b$ holds. We say an element $a \in L$ is *left modular* if aMx holds for every $x \in L$. We say an element $b \in L$ is *right modular* if yMb holds for every $y \in L$. An element of L is called *modular* if it is both left modular and right modular. In a semimodular lattice L, since for all $a, b \in L$, aMb implies bMa (Theorem 1.9.18 in [3, p.67]), to define a modular element, it is enough to define the element to be left modular. In an

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atomistic lattice, $\hat{0}$ and $\hat{1}$ are trivial modular elements. In a geometric lattice L, elements $\hat{0}$, $\hat{1}$ and atoms are trivial modular elements.

We say an element $d \in L$ is *distributive* if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ holds for all $x, y \in L$. The dual of distributive elements are called *dually distributive elements.* Clearly, a dually distributive element is right modular. We say an element $d \in L$ is *standard* if $x \wedge (d \vee y) = (x \wedge d) \vee (x \wedge y)$ holds for all $x, y \in L$. Clearly, a standard element is left modular. We say an element $d \in L$ is *neutral* if d is both standard and dually distributive. Clearly, a neutral element is modular. Obviously, $\hat{0}$ and $\hat{1}$ are trivial neutral elements of L. A lattice is called *distributive* if each of its elements is distributive. A equivalent condition for a lattice being distributive is that all of its elements are dually distributive (see Theorem 9 in [1, p.11]). It is well known that any distributive lattices are modular. A *Boolean* lattice is defined to be a distributive, geometric lattice.

A *cutset* of L is a subset of L which meets every maximal chain in L. A M-cutset or N-cutset is a cutset consisting of modular elements or neutral elements in L, respectively. In two previous papers, the author proved that an atomistic lattice of finite length is a geometric lattice if it has a M-cutset in which $\hat{0}$, $\hat{1}$ are the trivial modular elements ([5]), and proved that a geometric lattice is a modular lattice if it has a M-cutset in which $\hat{0}, \hat{1}$ and atoms are trivial modular elements ([4]). In this paper the author will continue to investigate the N-cutset condition for distributive lattices. Since every element of a distributive lattice is neutral, and in some sense, modular lattices as a kind of direct generalization of distributive lattices have more close relationship with distributive lattices than other lattices, it looks more natural to investigate the N-cutset condition on modular geometric lattices. But, generally, only $\hat{0}$ and $\hat{1}$ are trivial neutral elements in both geometric lattices and modular geometric lattices, the author find it has no essential difference to apply the N-cutset condition on geometric lattices from doing that on modular geometric lattices. The author shall prove that a geometric lattice L is distributive or Boolean if L processes a N-cutset in which $\hat{0}$ and $\hat{1}$ are trivial neutral elements. By the main result in [5], this result is generalized to atomistic lattices of finite length. In the next section, we shall present some necessary definitions and lemmas. The main results in this paper will be proved in Section 3.

1.1. **Preliminaries.**

Given two elements $a, b \in L$ with $a \leq b$, define the *interval* $[a, b] = \{c \in L : a \leq c \leq b\}$. An element x of a lattice L is *complemented* if there is an element $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. Note that a neutral element has at most one complement and any such complement is neutral (Theorem 12 [1, p.69]). An element is *uniquely* complemented if it has only one complement. A lattice L is called *complemented* if every element of L is complemented. A geometric lattice L always is complemented. Note that an interval of a geometric lattice is again a geometric lattice. Hence each element x in an interval $[a, b]$ of a geometric lattice is complemented in $[a, b]$, i.e., there is an element $y \in [a, b]$ such that $x \vee y = b$ and $x \wedge y = a$. Clearly, a neutral element in a geometric lattice is uniquely complemented and its complement is neutral.

An element $a \in L$ is called *separating* if for $x, y \in L$, $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ together imply $x = y$. The first lemma is Theorem 2.2.3 in [3, p.76] or partial of Theorem 4 in [2, p.184].

Lemma 1.1. *An element is neutral if and only if it is distributive, dual distributive, and separating.*

The following lemma is partial of Theorem 9 in [2, p.188].

Lemma 1.2. *Both the join and the meet of any two neutral elements are neutral*

From the proceeding lemma 1.2, it follows that an atomistic lattice L is Boolean if and only if each of its atoms is neutral. The next lemma is Theorem 3.1 in [5].

Lemma 1.3. *An atomistic lattice of finite length is semimodular if L has a M-cutset which contains no $\hat{0}$ and $\hat{1}$.*

1.2. **Main results.**

Theorem 1.4. *Let L be a geometric lattice. If every maximal chain of L contains a neutral element different from $\hat{0}$ and $\hat{1}$, then L is a Boolean lattice.*

Proof. By the definition of distributive lattices, we need prove each element of L is distributive. By Lemma 1.1, it suffices to prove each element of L is neutral. Since L is atomistic, by Lemma 1.2, it suffices to prove each atom of L is neutral. It's easily seem the theorem is true if $\ell(L) = 2$. Suppose $\ell(L) = n\ (> 2)$ and assume the theorem is true for every lattice of rank $< n$.

Let $y_0 \in L$ be an atom. Suppose y_0 isn't neutral. Then the interval $[y_0, \hat{1}]$ as a geometric lattice satisfies the condition and by induction, it is a Boolean lattice. Let x_1 be one minimal neutral element of L in a maximal chain of $[y_0, \hat{1}]$. Then there is an element y_1 which is the unique complement of x_1 in $[y_0, \hat{1}]$. For each element $y \in [y_0, y_1]$, from $y_0 \leq x_1 \wedge y \leq x_1 \wedge y_1 = y_0$ it follows $x_1 \wedge y = y_0$. Since x_1 is neutral and y_0 isn't neutral, by Lemma 1.2, any element in $[y_0, y_1]$ isn't a neutral element of L. Then by the condition, each maximal chain in $[y_1, \hat{1}]$ has a neutral element ($< \hat{1}$) of L. Similarly, all of the intervals $[\hat{0}, y_0], [y_0, y_1], \ldots, [y_{k-1}, y_k]$ $(k \geq 1)$ contain no neutral elements of L, where each y_i is the complement of the neutral element of L in the interval $[y_{i-1}, \hat{1}]$, said x_i, satisfying $x_i \wedge y_i = y_{i-1}$ and $x_i \vee y_i = \hat{1}$ for $1 \leq i \leq k$. By the condition, each maximal chain in $[y_k, \hat{1}]$ has a neutral element $(< \hat{1})$ of L. Since the length of L is finite, it implies $y_k = \hat{1}$ for some index k. Then a desired maximal chain in $[\hat{0}, y_0] \cup [y_0, y_1] \cup \cdots \cup [y_{k-1}, \hat{1}]$ contains no neutral elements of L different from $\hat{0}$ and $\hat{1}$, which yields a contradiction. The contradiction shows L is a Boolean lattice. □

Since a neutral element is modular, by Lemma 1.3 and Theorem 1.4, we immediately have the following corollary.

Corollary 1.5. *Let L be an atomistic lattice of finite length. If every maximal chain of L contains a neutral element different from $\hat{0}$ and $\hat{1}$, then L is a Boolean lattice.*

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